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THE COMPLEX OF AXES OF A CENTRAL QUADRIC SURFACE.

BY EDWARD V. HUNTINGTON.

The complex of axes of a quadric surface was first presented by Reye, in 1868, in his *Geometrie der Lage.**

The analytic equations were first stated by Reye.† A detailed discussion of these (unsymmetrical) equations is given by E. Schilke.‡

Additions to the theory have been made by M. G. Koenigs,§ E. Waelsch‖ and H. E. Timerding.¶

The object of the present paper is to restate as briefly as possible such of the chief results of this small chapter of modern geometry as can be conveniently developed by elementary methods.

1. DEFINITION. *A line through any point, perpendicular to the polar plane of that point with respect to a quadric surface, is called an* AXIS *of the quadric.*** The point, P, is the *pole* of the axis, and the polar plane of P is the *polar plane* of the axis. The *foot* of the axis is the point where it meets its polar plane. All the axes of a quadric taken together constitute the *complex of axes* (*Axencomplex*) of that quadric.

Every point in space (except the centre and the infinite points on the axes of symmetry) is the pole of only one axis; and since there are ∞^3 points in space, there will be in general not more than ∞^3 axes of any quadric. But there are ∞^4 lines in space; hence it is evident that not every line can be an axis.

Our task is, then, to ascertain what lines are axes and what sort of a figure the entire complex of axes forms in space. We shall confine ourselves to the central quadrics, represented by

$$\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}=1 \tag{1}$$

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* See Vortrag XV, Vol. 2, of the third edition (1892).

† *Annali di Matematica*, Ser. 2, Vol 2, p. 1–12 (1868–69).

‡ *Zeitschrift für Math. u. Phys.* (*Schlömilch*), Vol. 19, p. 550–564 (1874).

‖ *Nouvelles Annales de Math.*, Ser. 3, Vol. 2, p. 267–272 (1883).

§ *Wiener Sitzungsberichte*, Vol. 95, 2. Abth., p. 549–578 (1887).

¶ *Annali di Matematica*, Ser. 3, Vol. 2, p. 239–261 (1899).

**For another definition see Theorem XIV.

Since the surfaces of revolution possess no interest in this connection, we shall assume that A, B, and C are all different. (*Cf.* Prob. 1.)

We add for reference the familiar equation of the polar plane of any point (x_1, y_1, z_1) with respect to (1):

$$\frac{x_1 x}{A}+\frac{y_1 y}{B}+\frac{z_1 z}{C}=1. \tag{2}$$

I. THE AXES, WITH THEIR POLES.

2. Some of the lines which belong to the complex may be discovered at once from geometrical considerations.

Thus, if the point P is taken on the quadric, its polar plane coincides with the tangent plane at P and the corresponding axis is perpendicular to the tangent plane at that point. Hence,

THEOREM I. *Every normal to the quadric is an axis, having its foot as pole.*

Again, if P is taken at the centre, its polar plane becomes the plane at infinity, to which all lines through the centre are perpendicular. Hence,

THEOREM II. *Every diameter of the quadric is an axis, having the centre as pole.*

Again, if P recedes to infinity in the direction of one of the axes of symmetry, its polar plane becomes the plane of symmetry perpendicular to that axis. Hence,

THEOREM III. *Every line perpendicular to a plane of symmetry is an axis, having its infinitely distant point as pole.*

We may show, further:

THEOREM IV. *Every line lying in a plane of symmetry is an axis.*

For, let a be any line in a plane of symmetry, and let k be the conic section in which the plane cuts the quadric surface.

Now the polar plane of any point in the plane of symmetry with respect to the quadric will be perpendicular to the plane of symmetry, and its trace in that plane will be the polar of the point with respect to the conic section k.

Through the pole Q of a with respect to k draw a line c perpendicular to a. The pole P of c with respect to k will lie on a (since c passes through the pole of a); and the polar plane of P with respect to the quadric will contain c and be perpendicular to a.

Therefore, a is an axis of the quadric; P is its pole, and c is the trace of its polar plane. The theorem is thus established.

We have thus far tacitly assumed that every axis has but a single pole; and it is not hard to show that this is in general the fact. Only in the case of an axis of symmetry, the polar plane of *any* point on the axis is perpendicular to the axis. Hence,

Theorem V. *The pole of an axis of symmetry is indeterminate.*

3. Let us now find the general condition under which any given line

$$\frac{x-\xi}{l}=\frac{y-\eta}{m}=\frac{z-\zeta}{n} \tag{3}$$

shall be an axis of the surface (1).

Suppose that the line (3) is an axis, and let P_1 (x_1, y_1, z_1) be its pole.

Then, since P_1 must lie on the line (3),

$$\frac{x_1-\xi}{l}=\frac{y_1-\eta}{m}=\frac{z_1-\zeta}{n}\,; \tag{4}$$

and since (3) must be perpendicular to the polar plane (2) of P_1,

$$\frac{x_1}{Al}=\frac{y_1}{Bm}=\frac{z_1}{Cn}\,. \tag{5}$$

Eliminating x_1, y_1, z_1 from these four equations, we find

$$(B-C)\frac{\xi}{l}+(C-A)\frac{\eta}{m}+(A-B)\frac{\zeta}{n}=0, \tag{6}$$

as the condition under which the line (3) is an axis of the quadric (1).

The equations of the present article give us readily the analytic proof for Theorems II–V (including the statement that every axis except the axes of symmetry has a single determinate pole), and also lead at once to the important theorems of the two following articles.

4. Suppose the direction cosines l, m, n of an axis (3) are given; then the coordinates (ξ, η, ζ) of any point of the axis must satisfy (6), and the coordinates (x_1, y_1, z_1) of the pole of the axis must satisfy (5). These equations are all of the first degree in the variables, and lack a constant term. (The only troublesome case occurs when two of the given cosines are 0, which is the case already considered in Th. III.) Hence, THEOREM VI. *All axes parallel to a given direction (except those perpendicular to a plane of symmetry) lie in a diametral plane, and their poles lie in a diameter.* (*Cf.* Prob. 7.)

5. Suppose, on the other hand, that a point (ξ, η, ζ) of the axis (3) is given; then the direction cosines l, m, n of the axis must satisfy (6), and the coordinates (x_1, y_1, z_1) of the pole of the axis must satisfy (4) and (5).

Dividing each term of (6) by one of the equal fractions in (3), we have

$$(B-C)\frac{\xi}{x-\xi}+(C-A)\frac{\eta}{y-\eta}+(A-B)\frac{\zeta}{z-\zeta}=0, \tag{7}$$

for the locus of all the axes that pass through the given point (ξ, η, ζ).

And similarly, dividing each term in (4) by one of the equal fractions in (5), we have, dropping the subscripts,

$$A\frac{x-\xi}{x}=B\frac{y-\eta}{y}=C\frac{z-\zeta}{z}, \tag{8}$$

for the locus of the poles of these axes.

We proceed to investigate the nature of both of these loci.

Examining the first of the equations, we see at once that (7) represents a cone of the second degree, with its vertex at the given point (ξ, η, ζ); it contains the diameter through the given point (*cf.* Th. II), and the lines $x=\xi$, $y=\eta$; $y=\eta$, $z=\zeta$; and $z=\zeta$, $x=\xi$ (*cf.* Th. III); and is called the *cone of axes* (*Axenkegel*) of the given point.

Since the cone contains a set of three mutually perpendicular generators it will be an *equilateral* cone, containing an infinite number of such sets.

The second locus, (8), is a space curve containing the poles of all axes through the given point, and therefore called the *pole-curve* (*Polcurve*) of that point.

To determine the order of the curve, set each of the equal fractions in (8) equal to a variable λ; we have then

$$x=\frac{A\xi}{A-\lambda},\quad y=\frac{B\eta}{B-\lambda},\quad z=\frac{C\zeta}{C-\lambda}, \tag{8a}$$

as the coordinates of any point of the pole-curve, in terms of the parameter λ. Substituting these values in the equation of any plane, $ax+by+cz+d=0$. we obtain a cubic equation for λ; and to each of the three values of λ thus determined corresponds one point of intersection of the curve with the plane. In other words, the curve is of the third order.

It is obvious that the pole-curve must lie wholly on the cone of axes; and this appears analytically from the fact that the coordinates (8*a*) of any point of the pole-curve satisfy identically equation (7).

The curve contains the centre (*cf.* Th. II), and the points at infinity on the lines $x=\xi$, $y=\eta$; $y=\eta$, $z=\zeta$; and $z=\zeta$, $x=\xi$ (*cf.* Th. III).

The results of this article may be stated as

THEOREM VII. *All axes through a given point lie on an equilateral quadric cone, having its vertex at the given point, and containing, besides the diameter through that point, the perpendiculars from that point to the planes of symmetry; and their poles form on this cone a space curve of the third order, passing through the given point, and containing, besides the centre, the points at infinity on the perpendiculars to the planes of symmetry.* (*Cf.* Prob. 5.)

6. A simple special case of Th. VII occurs when the given point is taken in a plane of symmetry.

Thus, if the point is $(\xi, \eta, 0)$, the cone (7) breaks up into the plane $z=0$ (*cf.* Th. IV), and the plane $\xi\,(B-C)\,(y-\eta)+\eta\,(C-A)\,(x-\xi)=0$, which is perpendicular to $z=0$; while the curve (8) breaks up into the equilateral hyperbola $(A-B)\,xy+B\eta x-A\xi y=0$ in the plane $z=0$, and the straight line $(A-C)\,x=A\xi$, $(B-C)\,y=B\eta$, perpendicular to that plane. Hence,

COR. I. *All axes lying in a plane of symmetry and passing through a given point (not in an axis of symmetry) have their poles on an equilateral hyperbola, which contains the given point and the centre, and has asymptotes parallel to the axes of symmetry in that plane.*

COR. II. *All axes that pierce a plane of symmetry in a given point (not in an axis of symmetry) lie in a plane perpendicular to the plane of symmetry; and their poles lie in a straight line, also perpendicular to the plane of symmetry.* (*Cf.* Prob. 6.)

7. The equations of the axis whose pole is the given point (x_1, y_1, z_1) are

$$A\frac{x-x_1}{x_1}=B\frac{y-y_1}{y_1}=C\frac{z-z_1}{z_1}; \qquad (9)$$

for, this line (9) contains the given point, and is perpendicular to the polar plane (2) of that point.

8. Two theorems follow immediately from these equations.

Thus, we know that the common value of the equal fractions in (9) is proportional to the distance along the line from the fixed pole to the variable point (x, y, z). Putting successfully $x=0$, $y=0$, $z=0$, we find the distances along the line from the pole to the three coordinate planes are proportional to $-A$, $-B$, $-C$. Hence,

THEOREM VIII. *The pole of an axis divides in a constant ratio the portion of the axis intercepted between two planes of symmetry.*

Again, we find that the values of x in (9) corresponding to $y=0$ and $z=0$ are in the ratio $A-B : A-C$. Hence,

THEOREM IX. *The distances from one plane of symmetry to the points where any axis cuts the other two, stand in a constant ratio.*

9. These two theorems enable us to state the following construction:

THEOREM X. *If the three planes of symmetry and any one axis (not perpendicular to or lying in a plane of symmetry), with its pole, are given, then all the other axes of the complex are determined, and can be constructed.*

Thus: from the points where the given axis cuts the planes XY and XZ draw perpendiculars g and h to OX, and determine the ratio of the intercepts, $Og : Oh$.

If we now draw any line g' in XY, perpendicular to OX, and construct the line h' in XZ, perpendicular to OX, such that $Og' : Oh' = Og : Oh$, then all lines that meet both g' and h' will be axes, by Th. IX.

Thus all the axes that pierce any quadrant of the XY plane can be constructed. To these we must add all diameters and all lines perpendicular to or lying in a plane of symmetry (Th. II, III, IV), and the complex is complete.

If the pole of the given axis is known, the poles of all the other axes can be found by Th. VIII. (*Cf.* also, Th. VII, Cor. II.)

II. THE AXES, WITH THEIR POLAR PLANES.

10. If (x_1, y_1, z_1) is the pole of an axis, then its polar plane is given by equation (2).

Now when the axis is made to pass through a fixed point, the coordinates of its pole are given in (8a); substituting these values for x_1, y_1, z_1 in (2), we have:

$$\frac{\xi x}{A-\lambda} + \frac{\eta y}{B-\lambda} + \frac{\zeta z}{C-\lambda} = 1, \tag{10}$$

(where λ is a variable parameter) as the equation of the polar plane of any axis passing through the given point (ξ, η, ζ).

This plane (10) is evidently the polar plane of the given point (ξ, η, ζ) with respect to the quadric whose equation is:

$$\frac{x^2}{A-\lambda}+\frac{y^2}{B-\lambda}+\frac{z^2}{C-\lambda}=1, \tag{11}$$

and we know that this equation represents a family of quadrics confocal to the given quadric (1), their principal sections being confocal conics. Hence,

THEOREM XI. *The polar planes of axes through a given point are the polar planes of that point with respect to a family of quadrics confocal to the given quadric.*

It is shown in the theory of confocal quadrics that three of these surfaces pass through any given point, and cut orthogonally at that point.

Consider then, the three confocal quadrics that contain the given point (ξ, η, ζ); the polar planes of (ξ, η, ζ) with respect to these three surfaces will be the three mutually perpendicular tangent places at that point; and the axes to which these polar planes belong are therefore the three normals to the confocal quadrics at the given point. Hence,

COR. I. *Every point in space is the foot of three axes, which meet at right angles at that point.*

11. It is shown, further, that if from any point P tangents are drawn to the quadric, the axes of symmetry of the cone thus formed are normals to the three confocal quadrics that pass through P. Hence,

COR. II. *The axes of symmetry of any circumscribing cone are axes of the quadric, the foot of each coinciding with the vertex of the cone.*

12. If $\lambda = A$, in the equation (11) of a family of confocal quadrics, the quadric becomes the conic section

$$x=0, \qquad \frac{y^2}{B-A}+\frac{z^2}{C-A}=1; \tag{12}$$

and when $\lambda = B$ and $\lambda = C$, we obtain corresponding curves in the planes $y=0$, $z=0$, their equations being found by advancing the letters.

These three curves, which belong to the family of confocal quadrics, are called the *focal curves* of that family.

Now when $x=0$ in the equations of an axis (9), we have the point

$$x=0, \quad y=\frac{B-A}{B}y_1, \quad z=\frac{C-A}{C}z_1;$$

and when $x=0$ in the equation (2) of the polar plane of that axis, we have the line

$$x=0, \quad \frac{y_1y}{B}+\frac{z_1z}{C}=1;$$

and these are evidently pole and polar with respect to the focal curve (12). Hence,

THEOREM XII. *The point where an axis cuts a plane of symmetry, and the line in which its polar plane cuts the same plane of symmetry, are pole and polar with respect to a focal curve of the quadric.*

Let the point P and the line a be pole and polar with respect to a focal curve of the quadric. Then every axis that pierces the given plane at P must be perpendicular to some plane containing a. Hence,

COR. *The feet of all axes that pierce a given plane of symmetry in a given point (not in an axis of symmetry) lie on a circle whose plane is perpendicular to the given plane, and whose diameter is the distance from the given point to its polar line with respect to that focal curve which lies in the given plane.* (*Cf.* Th. VII, Cor. II.)

13. We can now state a second construction for the complex of axes, analogous to that given in Th. X. Thus:

THEOREM XIII. *If one of the focal curves of the quadric is given, all the axes of the complex, with their feet, are determined and can be constructed.*

For, the feet of all the axes that lie in the given plane can be constructed as in Th. IV; and all the axes that pierce the given plane at any point (except the centre) can be constructed, with their feet, by Th. XII, Cor. The only axes that remain are those described in Th. II and Th. III.

III. RECIPROCAL POLARS.

14. We know that, if a point moves in a given plane, its polar plane will revolve about a fixed point. Hence, if a point moves along a given *line* (common to two planes), its polar plane will revolve about a second *line* (joining two points); and conversely, if a plane revolves about a given line, its pole will move along a second line.

Two lines so related are called reciprocal polar lines, or simply, *reciprocal polars;* the polar plane of every point on either contains the other.

The reciprocal polar of any axis will be perpendicular to that axis. For it lies in the polar plane of the axis, and all lines in that plane are perpendicular to the axis.

Conversely, any line a, perpendicular to its reciprocal polar, b, will be an axis. For, a plane containing b and the common perpendicular of a and b will be perpendicular to a, and its pole will lie on a; hence, a is an axis.

In the synthetic treatment of the complex of axes, this property is conveniently used as the definition of an axis; thus:

THEOREM XIV. *A necessary and sufficient condition that a line shall be an axis is that it be perpendicular to its reciprocal polar.*

15. THEOREM XV. *The axes of symmetry of any conic section traced on the quadric are axes of the quadric.**

For, let a be an axis of symmetry of such a conic. The tangent planes to the quadric at the extremities of a will contain the parallel tangent lines to the conic at these points; hence the line of intersection of these tangent planes, which is the reciprocal polar of a, will be parallel to both the tangent lines, and therefore perpendicular to a. Thus a is an axis, by Th. XIV.

16. THEOREM XVI. *If two axes lie in a plane, the line joining their poles is also an axis.*

For, the polar planes of each of the poles, P, P', will be perpendicular to the corresponding axis, and hence to the plane, α, in which the axes lie. The line of intersection of these polar planes, which is the reciprocal polar of PP', will then be perpendicular to α, and hence to the line PP' which lies in α. Therefore PP' is an axis, by Th. XIV.

The present theorem may also be stated thus: *All chords of the pole-curve of any point* (Th. VII) *are axes of the quadric.*

COR. I. *The line joining the feet of any two normals that lie in the same plane is an axis of the quadric.*

17. THEOREM XVII. *The product of the distances from the poles of two reciprocal polar axes from a plane of symmetry is constant.*

An outline of the analytic proof for this theorem is as follows.

The axis whose pole is P_1 (x_1, y_1, z_1) is represented by (9), and its reciprocal polar will be the intersection of the polar planes of any two points whose coordinates satisfy (9).

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*It was from this point of view that the axes were first studied; *cf.* Th. XI, Cor. II.

Selecting the points (x_1, y_1, z_1) and $(x_1/A+x_1, y_1/B+y_1, z_1/C+z_1)$ of the line (9), we can use

$$\frac{x_1x}{A}+\frac{y_1y}{B}+\frac{z_1z}{C}=1, \qquad \frac{x_1x}{A^2}+\frac{y_1y}{B^2}+\frac{z_1z}{C^2}=0$$

as the equations of the reciprocal polar axis.

Let P_2 be the pole of this axis. Then P_2 must lie on the axis and the polar plane of P_2 must be perpendicular to the axis, and hence to both the planes which determine the axis. Hence,

$$\frac{x_1x_2}{A}+\frac{y_1y_2}{B}+\frac{z_1z_2}{C}=1, \quad \frac{x_1x_2}{A^2}+\frac{y_1y_2}{B^2}+\frac{z_1z_2}{C^2}=0, \quad \frac{x_1x_2}{A^3}+\frac{y_1y_2}{B^3}+\frac{z_1z_2}{C^3}=0.$$

Solving these three linear equations for x_1x_2, y_1y_2, z_1z_2, we find,

$$x_1x_2=a, \quad y_1y_2=b, \quad z_1z_2=c; \tag{13}$$

where

$$a=\frac{A^3}{(A-B)(A-C)}, \quad b=\frac{B^3}{(B-C)(B-A)}, \quad c=\frac{C^3}{(C-A)(C-B)}.$$

18. The following corollaries follow at once from (13).

Given, the poles of two reciprocal polar axes; then:

COR. I. *If one moves in a plane parallel to a plane of symmetry, the other will move in a parallel plane.*

COR. II. *If one moves in a plane containing an axis of symmetry, the other will move in a plane containing the same axis of symmetry.*

COR. III. *If one moves in any diametral plane, the other will in general move in a quadric cone.*

COR. IV. *If one moves in any plane whatever, the other will move in general in a surface of the third degree.*

COR. V. *If one moves in a diameter, the other will move in a diameter.*

COR. VI. *If one moves in a line parallel to a principal axis, the other will move in a parallel line.*

IV. COAXIAL QUADRICS.

19. In conclusion, let us see what changes in the given quadric (1) can be made without disturbing the complex of axes of that quadric; in other words, let us see whether the same complex may not belong to more than one quadric.

The condition (6), under which a given line is an axis of the quadric (1), remains unaltered when we add any constant to A, B, and C, or when we multiply A, B, and C by any constant factor.

That is, any line which is an axis of (1) will be also an axis of the quadric represented by

$$\frac{x^2}{A-\lambda}+\frac{y^2}{B-\lambda}+\frac{z^2}{C-\lambda}=\mu \qquad (14)$$

whatever the values of the constants λ and μ.

This equation contains two arbitrary parameters, and therefore represents a doubly infinite series of quadrics.

If λ alone varies, while $\mu=1$, we have a family of quadrics confocal to (1).

If, for any given value of λ, μ is allowed to vary, we have a family of quadrics similar, concentric, and similarly placed to the quadric which corresponds to the given value of λ. If we use the word *concyclic* to describe quadrics that are similar, concentric, and similarly placed, we have the theorem:

THEOREM XVIII. *All the quadrics of a confocal family with all the quadrics concyclic to each of these quadrics have the same complex of axes.*

The quadrics represented by (14) have therefore been called by Reye *coaxial*. All coaxial quadrics have the same axes, but these axes do not in general have the same poles and the same feet with respect to all the quadrics.

If $\lambda=0$, and only μ be allowed to vary, then not only the condition (6), but also the locus (9) of the poles of axes through a given point, will remain unaltered. Hence,

COR. I. *All concyclic quadrics not only have the same complex of axes, but each axis preserves its own pole.*

If $\mu=1$, and only λ be allowed to vary, we have a family of confocal quadrics, which is wholly determined if one of its focal curves is known. Hence, by Th. XIII,

COR. II. *All confocal quadrics not only have the same complex of axes, but each axis preserves its own foot.*

MISCELLANEOUS PROBLEMS.

1. If the quadric (1) is a quadric of revolution, the complex of axes is composed of: (a) all lines perpendicular to the axis of rotation; and (b) all

lines meeting that axis or parallel to it. In the case of a sphere, every line in space may be regarded as an axis. (Th. III.)

2. The poles of all axes that meet a given line lie on a quadric surface containing the given line. If the given line is a diameter, the surface becomes a cone containing, besides the given line, the three axes of symmetry.

Let the given line be $ax + by + cz + d = 0, \quad a'x + b'y + c'z + d' = 0.$ From (9) express the coordinates of any point of the axis in terms of a variable parameter. Then

$$\frac{ax + by + cz + d}{\frac{ax}{A} + \frac{by}{B} + \frac{cz}{C}} = \frac{a'x + b'y + c'z + d'}{\frac{a'x}{A} + \frac{b'y}{B} + \frac{c'z}{C}}$$

is the equation of the required quadric.

3. The poles of all axes that meet a plane of symmetry along a line perpendicular to an axis of symmetry lie in a plane perpendicular to that axis. (*Cf.* Th. VII, Cor. II, and Prob. 6, below.)

(Special case of the preceding.)

4. The poles of all axes parallel to a given plane lie in a diametral plane; the poles of all axes in a given plane lie on a line.

Let the given plane be $ax + by + cz + d = 0$; then, using (9),

$$\frac{ax}{A} + \frac{by}{B} + \frac{cz}{C} = 0$$

is the equation of the required diametral plane.

5. The feet of all axes through a given point form on the cone of axes of that point (Th. VII) a space curve of the fifth order,* containing the infinite point of the diameter through the given point, and the feet of the perpendiculars from the given point to the planes of symmetry; and passing three times through the given point.

In this and the following problems on the feet of the axes, it is convenient to use the unsymmetrical equations of a line. Show that the equations

$$x = \frac{y}{m} + b\,, \quad x = \frac{z}{n} + kb, \qquad \text{where } k = \frac{A - C}{A - B},$$

represent any axis of the quadric (1); find the pole of this axis, and show that

$$x + my + nz = \frac{A - B}{b}$$

is the equation of its polar plane.

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* This curve has been studied in detail by A. Schmitz: *Programm der K. bayr. Studienanstalt, Neuberg a. D.*, for 1886–87.

Expressing the condition that the axis shall pass through the given point (ξ, η, ζ) and eliminating m, n, and b, we find

$$x(x-\xi)+y(y-\eta)+z(z-\zeta)=(A-B)\frac{(x-\xi)(y-\eta)}{\xi y-\eta x}=(A-C)\frac{(x-\xi)(z-\zeta)}{\xi z-\zeta x}$$

as the equations of the required locus.

6. The feet of all axes through a given point in a plane of symmetry (but not in an axis of symmetry) lie (*a*) in a curve of the third order in that plane; and (*b*) in a circle perpendicular to that plane and bisected by it. (*Cf.* Th. VII, Cor. I, II.)

(Special case of the preceding.)

7. The feet of all axes parallel to a given direction (except those perpendicular to a plane of symmetry) lie in an equilateral hyperbola. (*Cf.* Th. VI.)

(Special case of Prob. 5 when the given point recedes to infinity in a given direction. There is also a curve of the third order, lying wholly at infinity, that belongs to the locus.)

8. The feet of all axes that meet a plane of symmetry along a line perpendicular to an axis of symmetry lie on a surface of the third order. The surface has two planes of symmetry and contains two families of circles; it is a special case of the Cyclide of Dupin. (*Cf.* Th. XII, Cor., and Prob. 3.)

If the given line is $y=0$, $x=b$, then

$$x+\frac{y^2}{x-b}+\frac{z^2}{x-kb}=\frac{A-B}{b}, \qquad \text{where } k=\frac{A-C}{A-B},$$

is the equation of the surface.

9. The six normals that can be drawn from a given point to a quadric surface lie on an equilateral quadric cone, containing, besides the diameter through that point, the perpendiculars from that point to the three planes of symmetry; and their feet lie on a space curve of the third order, containing, besides the centre and the given point, the points at infinity on the perpendiculars to the planes of symmetry. (Th. I, Th. VII.) (Steiner.)

10. Through any given point an infinite number of coaxial central quadrics can be passed. Their normals at this point lie on a quadric cone.

11. The polar planes of a given point with respect to a family of concyclic quadrics are parallel. (Th. XVIII, Cor. I.)

12. The poles of a given plane with respect to a family of confocal quadrics lie in a line perpendicular to that plane. (Th. XVIII, Cor. II.)

13. A given plane cuts a family of confocal quadrics in a series of conic sections whose axes of symmetry are tangent to a fixed parabola and whose

centres lie on a fixed line; this line is the directrix of the parabola; it passes through the foot of the perpendicular from the centre on the given plane, and through the foot of the perpendicular (Prob. 12) in which lie the poles of the given plane with respect to the quadrics.

14. All axes in a given plane envelop a parabola; their poles lie on a line tangent to the parabola; and their feet lie on a curve of the third order. The pole of this line with respect to the conic in which the given plane cuts the surface is a double point of the curve.

15. Discuss the complex of axes of a paraboloid.

STRASSBURG, GERMANY, JANUARY, 1900.